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The Kronecker Product of S_n-Characters and an $A \otimes B$ Theorem for Capelli Identities

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If A and B satisfy a Capelli identity then so does $A \otimes B$. To prove this we show that the height of $[\lambda] \otimes [\mu]$ is bounded by the product of the heights of $[\lambda]$ and $[\mu]$.

To state the main results we need the following (standard)

Notations. Write $\lambda = (\alpha_1, ..., \alpha_r) \in \mathrm{Par}(n) =$ the partitions of n, if $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ and $\alpha_1 + \cdots + \alpha_r = n$. If $\alpha_r \neq 0$, call $h(\lambda) = r$ the height of λ.

Assume throughout that F is a field of characteristic zero.

The irreducible characters of the symmetric group S_n are $\{[\lambda] \mid \lambda \in \mathrm{Par}(n)\}$, and $[\lambda]$ can be obtained from λ by the theory of Young tableaux [2]. Any S_n character χ can be written as $\chi = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda [\lambda]$. Write $h([\lambda]) = h(\lambda)$ and $h(\chi) = \max\{h(\lambda) \mid \lambda \in \mathrm{Par}(n)$ and $m_\lambda \neq 0\}$. Also define $\sum_{\lambda \in \mathrm{Par}(n)} m_\lambda [\lambda] \leqslant \sum_{\lambda \in \mathrm{Par}(n)} m'_\lambda [\lambda]$ if $m_\lambda \leqslant m'_\lambda$ for all $\lambda \in \mathrm{Par}(n)$.

In the (semi-simple) group algebra FS_n there is a unique minimal two-sided ideal I_λ that corresponds to each $\lambda \in \mathrm{Par}(n)$. A faithful left FS_n module M defines an S_n representation $\rho = \rho_M$ whose trace is the S_n character $\chi = \chi_M$. We say that χ_M is afforded by M. For example, $[\lambda]$ is afforded by any minimal left ideal $J_\lambda \subseteq I_\lambda$. If $\chi = \chi_M$ and $\Psi = \Psi_N$ are two S_n-characters then their inner (Kronecker) product $\chi \otimes \Psi$ is afforded by $M_F \otimes N$, whose FS_n module structure is determined by $\sigma(m \otimes n) = \sigma(m) \otimes \sigma(n)$, $\sigma \in S_n$, $m \in M$, $n \in N$.

The cocharacter sequence (c.c.s.) $\chi_n(A)$ of a *P.I.* algebras was introduced in [5] and is essential for this note. Finally, we call the non-commutative polynomial

$$d_{m+1}[x, y] = \sum_{\sigma \in S_{m+1}} \mathrm{sgn}(\sigma)\, x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_m x_{\sigma(m+1)}$$

a Capelli polynomial.

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Introduction. The following was proved in [6]:

THEOREM (a). *An F-algebra A satisfies* $d_{m+1}[x, y]$ *if and only if for all* n, $h(\chi_n(A)) \leqslant m$.

We apply it to prove

THEOREM (b). *If A satisfies* $d_{k+1}[x, y]$ *and B satisfies* $d_{l+1}[x, y]$ *then* $A \otimes B$ *satisfies* $d_{kl+1}[x, y]$.

The proof goes as follows. We first prove

THEOREM (c). *For any P.I. algebras* A, B, $\chi_n(A \otimes B) \leqslant \chi_n(A) \otimes \chi_n(B)$ (*the inner product of* $\chi_n(A)$ *and* $\chi_n(B)$).

This is a character interpretation of [4, Theorem 5.1]. We next apply a classical theorem of H. Weyl [2, Theorem 5.2] to prove

THEOREM (d). $h(\chi \otimes \Psi) \leqslant h(\chi) \cdot h(\Psi)$ *for any two* S_n*-characters* χ, Ψ. *Theorem* (b) *is now an obvious consequence of Theorems* (a), (c) *and* (d).

Some easy arguments about minimal Capelli identities for matrix algebras show that both Theorems (b) and (d) are the best possible.

Here are the details: Let $V_n = \operatorname{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ and identify $V_n \equiv FS_n$ by identifying $x_{\sigma(1)} \cdots x_{\sigma(n)} \equiv \sigma \in S_n$.

Let A, B be F algebras, $A \otimes B = A_F \otimes B$ and let $A \otimes B \supseteq T = \{a \otimes b \mid a \in A, b \in B\}$ and $T^n = \underbrace{T \times \cdots \times T}_{n}$, then

LEMMA 1. *The polynomial* $f(x_1, ..., x_n) \in V_n$ *is an identity for* $A \otimes B$ *if and only if* $f(t_1, ..., t_n) = 0$ *for all* $(t_1, ..., t_n) \in T^n$.

Proof. Follows immediately from the multilinearity of $f(x)$.

LEMMA 2. *Let* $\sigma, \eta \in S_n$ *and assume* $a_j \otimes b_j = a'_j \otimes b'_j$ *for* $j = 1, ..., n$, *then* $a_{\sigma(1)} \cdots a_{\sigma(n)} \otimes b_{\eta(1)} \cdots b_{\eta(n)} = a'_{\sigma(1)} \cdots a'_{\sigma(n)} \otimes b'_{\eta(1)} \cdots b'_{\eta(n)}$.

Proof. Any $\sigma \in S_n$ induces an automorphism φ_σ on $A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_{n}$, given by $\varphi_\sigma(a_1 \otimes \cdots \otimes a_n) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$. For $\eta \in S_n$ we have a similar automorphism $\varphi_\eta : B^{\otimes n} \to B^{\otimes n}$. Together they induce $\varphi_{\sigma,\eta} = \varphi_\sigma \otimes \varphi_\eta : A^{\otimes n} \otimes B^{\otimes n} \to A^{\otimes n} \otimes B^{\otimes n}$. Identify $A^{\otimes n} \otimes B^{\otimes n} \equiv (A \otimes B)^{\otimes n}$ and let $\pi((a_1 \otimes b_1) \otimes \cdots \otimes (a_n \otimes b_n)) = a_1 \cdots a_n \otimes b_1 \cdots b_n$; π maps $A^{\otimes n} \otimes B^{\otimes n}$ into $A \otimes B$. The proof follows since

$$\begin{aligned} a_{\sigma(1)} \cdots a_{\sigma(n)} \otimes b_{\eta(1)} \cdots b_{\eta(n)} &= \pi\varphi_{\sigma,\eta}((a_1 \otimes b_1) \otimes \cdots \otimes (a_n \otimes b_n)) \\ &= \pi\varphi_{\sigma,\eta}((a'_1 \otimes b'_1) \otimes \cdots \otimes (a'_n \otimes b'_n)) \\ &= a'_{\sigma(1)} \cdots a'_{\sigma(n)} \otimes b'_{\eta(1)} \cdots b'_{\eta(n)}. \quad \text{Q.E.D.} \end{aligned}$$

COROLLARY 5. *Let $\sigma, \eta \in S_n$. The map $\Psi_{\sigma,\eta}: T^n \to A \otimes B$, given by*

$$\Psi_{\sigma,\eta}(a_1 \otimes b_1, ..., a_n \otimes b_n) = a_{\sigma(1)} \cdots a_{\sigma(n)} \otimes b_{\eta(1)} \cdots b_{\eta(n)}$$

is well defined.

LEMMA 6. *The action of $V_n\,{}_F\!\otimes V_n$ on T^n given by*

$$(f \otimes g)(a_1 \otimes b_1, ..., a_n \otimes b_n) = f(a_1, ..., a_n) \otimes g(b_1, ..., b_n)$$

is well defined and maps T^n into $A \otimes B$.

Proof. Here $f = \sum_{\sigma \in S_n} \alpha_\sigma \sigma \equiv \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is viewed as a polynomial function, etc.

Let Fnc $= \{$all functions $f: T^n \to A \otimes B\}$. It clearly is an F-vector space and for $\sigma, \eta \in S_n$, $\Psi_{\sigma,\eta} \in$ Fnc. Map $V_n \times V_n \to$ Fnc by

$$\left(\sum_\sigma \alpha_\sigma \sigma, \sum_\eta \beta_n \eta\right) \to \sum_{\sigma,\eta} \alpha_\sigma \beta_n \Psi_{\sigma,\eta}$$

The bilinearity being trivial to check, we can now map $V_n \otimes V_n \to$ Fnc by

$$\left(\sum_\sigma \alpha_\sigma \sigma\right) \otimes \left(\sum_\eta \beta_\eta \eta\right) \to \sum_{\sigma,\eta} \alpha_0 \beta_n \Psi_{\sigma,\eta}$$

This defines an action of $V_n \otimes V_n$ on T^n which clearly has the above property. Q.E.D.

Map $V_n \to \bar{V}_n \subseteq V_n \otimes V_n$ by $\sigma \to \sigma \otimes \sigma = \bar{\sigma}$, $\sigma \in S_n$. Clearly $V_n \cong \bar{V}_n$. The action $\sigma(f \otimes g) = \bar{\sigma}(f \otimes g) = (\sigma \otimes \sigma)(f \otimes g) = \sigma f \otimes \sigma g$ makes $V_n \otimes V_n$ a left V_n module. If $M, N \subseteq V_n$ are left ideals with S_n-characters χ_M, χ_N, then $M\,{}_F\!\otimes N$ is a left V_n module whose character $\chi_{M \otimes N} = \chi_M \otimes \chi_N$ is their inner product. Note that the action of V_n on T^n, given by $\sigma(a_1 \otimes b_1, ..., a_n \otimes b_n) = a_{\sigma(1)} \cdots a_{\sigma(n)} \otimes b_{\sigma(1)} \cdots b_{\sigma(n)}$ coincides with the action of $\bar{V}_n$ on T^n as a subalgebra of $V_n \otimes V_n$.

Next, let A, B be two $P.I.$ algebras, $I(A) = Q$, $I(B) = P$ and $I(A \otimes B) = R$ their corresponding T-ideals of identities in the free algebra $F\langle x \rangle$. Write $Q_n = Q \cap V_n$, then $V_n = Q_n \oplus J_n$ for some left ideal J_n complementary to Q_n. Similarly $V_n = P_n \oplus K_n$ and $V_n = R_n \oplus L_n$.

THEOREM 7. *The V_n-left module L_n is isomorphic to a submodule of $J_n \otimes K_n$.*

Proof. Denote $(a \otimes b) = (a_1 \otimes b_1, ..., a_n \otimes b_n) \in T^n$ and let $I'_n = I'_n(A \otimes B) = \{k \in V_n \otimes V_n \mid k(a \otimes b) = 0$ for all $(a \otimes b) \in T^n\}$. Obviously, I'_n is a left V_n module (it is even a left $V_n \otimes V_n$ ideal). The isomorphism

$V_n \cong \bar{V}_n$ induces $V_n = R_n \oplus L_n \cong \bar{R}_n \oplus \bar{L}_n$. By lemma 1, $I'_n \cap \bar{L}_n = 0$. Both are V_n left submodules of $V_n \otimes V_n$. Let $\bar{L}_n \subseteq M'_n \subseteq V_n \otimes V_n$ be a left V_n submodule, maximal with respect to $M'_n \cap I'_n = 0$. Because V_n is completely reducible (char $F = 0$), $V_n \otimes V_n = I'_n \oplus M'_n$.

On the other hand, $V_n \otimes V_n = (Q_n \oplus J_n) \otimes (P_n \oplus K_n) = I''_n \oplus M''_n$, where $I''_n = (Q_n \otimes P_n) \oplus (Q_n \otimes K_n) \oplus (J_n \otimes P_n)$ and $M''_n = J_n \otimes K_n$. Trivially, $I''_n \subseteq I'_n$ hence, by complete reducibility, $I'_n = I''_n \oplus \tilde{M}_n$, so $V_n \otimes V_n = I''_n \oplus M''_n \oplus M'_n = I''_n \oplus M''_n$. Thus $\tilde{M}_n \oplus M'_n \cong M''_n$ so both submodules $\bar{L}_n \subseteq M'_n$ are isomorphic to submodules of $M''_n = J_n \otimes K_n$. Q.E.D.

Recall that $\sum_{\lambda \in \mathrm{Par}(n)} m_\lambda[\lambda] \leqslant \sum_{\lambda \in \mathrm{Par}(n)} m'_\lambda[\lambda]$ if for all λ, $m_\lambda \leqslant m'_\lambda$. Theorem 7 clearly implies

THEOREM 8. *Let $\chi_n(A)$, $\chi_n(B)$ and $\chi_n(A \otimes B)$ be the* c.c.s. *of the (P.I.) algebras A, B and $A \otimes B$, with codimensions $c_n(A)$, $c_n(B)$ and $c_n(A \otimes B)$. Then*

$$\chi_n(A \otimes B) \leqslant \chi_n(A) \otimes \chi_n(B) \qquad (\textit{inner product}).$$

In particular, $c_n(A \otimes B) \leqslant c_n(A) \cdot c_n(B)$.

Our next goal is to show that $h(\chi \otimes \Psi) \leqslant h(\chi) \cdot h(\Psi)$ for any two S_n characters. The following theorem of Weyl [2, Theorem 5.2] is needed:

THEOREM 9 (Weyl). *Let U be a vector space of finite dimension. Define $\varphi: FS_n \to \mathrm{End}(U^{\otimes n})$ by*

$$\varphi_\sigma(u_1 \otimes \cdots \otimes u_n) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

where $\sigma \in S_n$ and $\varphi_\sigma = \varphi(\sigma)$, then $\varphi(FS_n) \cong \bigoplus_{\lambda \in \mathrm{Par}(n), h(\lambda) \leqslant \dim U} I_\lambda$ as algebras.

COROLLARY 10. *Clearly, φ makes $U^{\otimes n}$ into an FS_n module. Let $M \subseteq U^{\otimes n}$ be an FS_n submodule and let $\lambda \in \mathrm{Par}(n)$ be any partition. If $I_\lambda \cdot M \neq 0$ then $h(\lambda) \leqslant \dim U$.*

Note. Call the action of FS_n on $U^{\otimes n}$, induced by φ, the canonical action. Corollary 10, which holds for that particular action, may be false for other actions. Note also that φ induces an action of FS_n on $\mathrm{End}(U^{\otimes n})$ since $\varphi(FS_n) \subseteq \mathrm{End}(U^{\otimes n})$. With these actions we can prove

LEMMA 11. *Let $J \subseteq \mathrm{End}(U^{\otimes n})$ be an FS_n left irreducible module. If $J \cong J_\lambda$ as left FS_n modules, then $h(\lambda) \leqslant \dim U$.*

Proof. By assumptions, $M = J \cdot U^{\otimes n} \subseteq U^{\otimes n}$ is an FS_n submodule of $U^{\otimes n}$ for the canonical action. Let $\Psi: J \to J_\lambda$ be an FS_n-module isomorphism. It is well known that J_λ contains an idempotent $0 \neq e = e^2$. Let $e' \in J$ satisfy $\Psi(e') = e$. Since Ψ is one-to-one and

$$\Psi(ee') = e\Psi(e') = e^2 = e = \Psi(e'),$$

therefore $ee' = e'$. Hence $J_\lambda M = J_\lambda J U^{\otimes n} \supseteq e' U^{\otimes n} \neq 0$ and by Corollary 10, $h(\lambda) \leqslant \dim U$. Q.E.D.

THEOREM 12. *Let $\chi_1 \otimes \chi_2$ be the inner product of the two S_n-characters χ_1 and χ_2, then*

$$h(\chi_1 \otimes \chi_2) \leqslant h(\chi_1) \cdot h(\chi_2)$$

Remark. In many cases $\leqslant$ is actually $\lneqq$ (see [3]). However, using some easy arguments about Capelli identities we later show that essentially Theorem 12 can not be improved.

Proof of the theorem. Clearly, enough to prove when $\chi_1 = [\lambda_1]$ and $\chi_2 = [\lambda_2]$ are irreducible: $\lambda_1, \lambda_2 \in \mathrm{Par}(n)$. Choose vector spaces U_1, U_2 with dimensions $\dim U_i = h(\lambda_i)$, $i = 1, 2$. The canonical actions induce the homomorphisms $\varphi_i : FS_n \to \mathrm{End}(U_i^{\otimes n})$, $i = 1, 2$, and by (Weyl's) Theorem 9, the restriction of φ_i to I_{λ_i} is an algebra isomorphism. Let $J_{\lambda_i} \subseteq I_{\lambda_i}$ be a minimal left ideal and write $J_i = \varphi_i(J_{\lambda_i})$, then $J_i \cong J_{\lambda_i}$ both as algebras and as FS_n left modules (via the action of FS_n on $\mathrm{End}(U_i^{\otimes n})$ induced by φ_i).

Let $W = U_1 \otimes U_2$ and identify $W^{\otimes n} \equiv U_1^{\otimes n} \otimes U_2^{\otimes n}$, then $J_1 \otimes J_2$ can naturally be identified with a subalgebra of $\mathrm{End}(W^{\otimes n})$: write $J_1 \otimes J_2 \subseteq \mathrm{End}(W^{\otimes n})$. Let $a_i \in \mathrm{End}(U_i^{\otimes n}) \cong (\mathrm{End}\, U_i)^{\otimes n}$, $i = 1, 2$. Assume $a_1 = T_1 \otimes \cdots \otimes T_n$, $T_j \in \mathrm{End}(U_1)$, $a_2 = S_1 \otimes \cdots \otimes S_n$, $S_j \in \mathrm{End}(U_2)$, and let $\sigma \in S_n$. Apply $\sigma a_1 \otimes \sigma a_2$ and $\sigma(a_1 \otimes a_2)$ to any generator $\mathbf{w}$ of $W^{\otimes n}$: $\mathbf{w} = (u_1 \otimes v_1) \otimes \cdots \otimes (u_n \otimes v_n) \equiv (u_1 \otimes \cdots \otimes u_n) \otimes (v_1 \otimes \cdots \otimes v_n)$, $(u_i \in U_1, v_j \in U_2)$ to conclude that for any $a_i \in \mathrm{End}(U_i^{\otimes n})$, $\sigma(a_1 \otimes a_2) = \sigma a_1 \otimes \sigma a_2$. In particular, $J_1 \otimes J_2$ is an FS_n module which affords the S_n character $[\lambda_1] \otimes [\lambda_2]$ (inner product).

By Lemma 11 and the definition of height,

$$h([\lambda_1] \otimes [\lambda_2]) \leqslant \dim W = (\dim U_1)(\dim U_2) = h(\lambda_1) \cdot h(\lambda_2). \quad \text{Q.E.D.}$$

We are now ready to prove

THEOREM 13. *Let A and B be two F-algebras satisfying the Capelli identities $d_{k+1}[x, y]$ and $d_{l+1}[x, y]$, respectively, then $A \otimes B$ satisfies $d_{kl+1}[x, y]$.*

Proof. By Theorem (a), $h(\chi_n(A)) \leqslant k$ and $h(\chi_n(B)) \leqslant l$. By Theorems 8 and 12, $h(\chi_n(A \otimes B)) \leqslant h(\chi_n(A) \otimes \chi_n(B)) \leqslant h(\chi_n(A)) \cdot h(\chi_n(B)) \leqslant kl$ so, again by Theorem (a), $A \otimes B$ satisfies $d_{kl+1}[x, y]$. Q.E.D.

We conclude by showing that Theorems 12 and 13 can not be improved. This depends on the fact that the algebra F_k of the $k \times k$ matrices over F satisfies $d_{k^2+1}[x, y]$ but not $d_{k^2}[x, y]$ (see [1]). Since $F_{kl} \cong F_k \otimes F_l$, we cannot have, in general, a Capelli identity of lower degree for $A \otimes B$. However the results of [3, 5, 6] imply that if A and B satisfy $d_3[x, y]$ then $A \otimes B$ satisfies $d_4[x, y]$.

Proposition 15 implies that Theorem 12 essentially can not be improved. Its proof requires the following lemma, which is due to Amitsur.

LEMMA 14. *If $n \geqslant 2k^2 - 1$ then $h(\chi_n(F_k)) = k^2$.*

Proof. Let $e = \sum_{\sigma \in S_{k^2}} (\operatorname{sgn} \sigma)\, x_{\sigma(1)} \cdots x_{\sigma(k^2)} \in FS_{k^2}$. It corresponds to the (one) column of height k^2. By the branching theorem,

$$(FS_n)\, e(FS_n) = \bigoplus_{\substack{\lambda \in \mathrm{Par}(n) \\ h(\lambda) \geqslant k^2}} I_\lambda .$$

By 6, $h(\chi_n(F_k)) \leqslant k^2$. If $h(\chi_n(F_k)) \lneqq k^2$, the elements of $(FS_n)\, e(FS_n)$ are all F_k identities. By an argument identical to that used in [6, 2(b)] deduce that $d_{k^2}[x_1,..., x_{k^2}; y_1,..., y_{k^2-1}] \cdot y_{k^2} \cdots y_{n-k^2-1}$ is an F_k identity, a contradiction. Q.E.D.

PROPOSITION 15. *Let k^2, l^2 be any squares and let $n \geqslant 2k^2 l^2 - 1$, then there are $\lambda_1, \lambda_2 \in \mathrm{Par}(n)$ such that $h(\lambda_1) = k^2$, $h(\lambda_2) = l^2$ and $h([\lambda_1] \otimes [\lambda_2]) = k^2 l^2$.*

Proof. By 14, $h(\chi_n(F_{kl})) = k^2 l^2$; i.e., there is some $\lambda \in \mathrm{Par}(n)$ having non-zero multiplicity in $\chi_n(F_{kl})$ and $h(\lambda) = k^2 l^2$. By 8

$$x_n(F_{kl}) = \chi_n(F_k \otimes F_l) \leqslant \chi_n(F_k) \otimes \chi_n(F_l)$$

so we must have some $[\lambda_1]$ in $\chi_n(F_k)$ and $[\lambda_2]$ in $\chi_n(F_l)$, both with non-zero multiplicity, such that $[\lambda]$ appears in $[\lambda_1] \otimes [\lambda_2]$. Since $h(\lambda_1) \leqslant k^2$, $h(\lambda_2) \leqslant l^2$ and $h(\lambda) = k^2 l^2$, we must have $h(\lambda_1) = k^2$ and $h(\lambda_2) = l^2$. Q.E.D.

We conjecture that Proposition 15 holds for any two heights—with an appropriate n.

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